J. Myanmar Acad. Arts Sci. 2018 **Vol. XVI. No.3**

GRAPHS WITH THE SPECIFIED EDGE GEODETIC NUMBERS

Kyaw Lin Aung[1], Tin Mar Htwe[2]

Abstract

Firstly we state some properties of the edge geodetic number of the connected graphs. Then the edge geodetic numbers of some special graphs are derived. Next we study the graphs with the edge geodetic number 2. We also state the necessary and sufficient conditions for a graph G with n vertices to have the edge geodetic number $g_e(G) = n - 1$ or $g_e(G) = n$. Finally we characterize the graphs which have the specified edge geodetic numbers.

Keywords: Edge geodetic cover, Edge geodetic basis, Edge geodetic number

1. Some Properties of the Edge Geodetic Number of a Connected Graph

In this section, being based on [1] through [4], we state the following basic results of the edge geodetic number of a connected graph.

1.1 Definitions.

An ***edge geodetic cover*** of a graph G is a set $S \subseteq V$ where V is the set of vertices of G such that every edge of G is contained in a geodesic joining some pair of vertices in S. The ***edge geodetic number*** $g_e(G)$ of G is the minimum order of its geodetic covers, and any edge geodetic cover of order $g_e(G)$ is an ***edge geodetic basis***. If S is an edge geodetic basis of G and a vertex $x \in S$, then x is called a ***basic vertex*** with respect to the basis S.

1.2 Example.

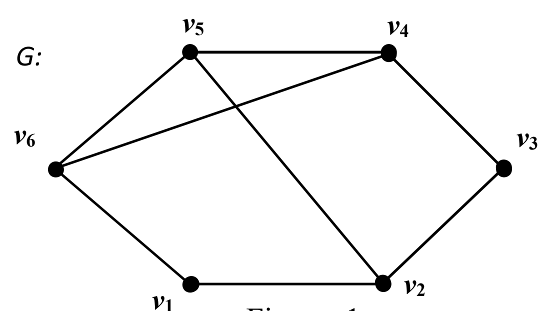


Figure. 1

1. Assistant Lecturer, Department of Mathematics, Sittway University,
2. Professor(Head), Department of Mathematics, Taungoo University

Consider the graph G shown in Fig. 1, any vertex set S containing two vertices from $\{ v_1, v_2, v_3, v_4, v_5, v_6\}$ is not an edge geodetic cover. Thus, $g_e(G) \geq 3$.

If $S = \{ v_1, v_3, v_5\}$, we can see that it is not an edge geodetic cover of G. But if $S = \{ v_2, v_4, v_6\}$, it is an edge geodetic cover of G and has minimum order. Thus the edge geodetic number $g_e(G)$ of G is 3 and S is an edge geodetic basis of G and v_2, v_4, v_6 are basic vertices with respect to S.

1.3 Remark. The edge geodetic number of a disconnected graph is the sum of the edge geodetic number of its components. Thus we will consider only connected graphs in the next sections.

1.4 Theorem. For every nontrivial graph G of order n, $2 \leq g_e(G) \leq n$.

Proof. An edge geodetic cover needs at least two vertices and therefore $g_e(G) \geq 2$. Clearly the set of all vertices of G is an edge geodetic cover of G. This means that $g_e(G) \leq n$. Thus $2 \leq g_e(G) \leq n$.

1.5 Theorem. Each extreme vertex of G belongs to every edge geodetic basis of G.

Proof. Let x be an extreme vertex of a connected graph G.

Suppose $W \subseteq V$ is any edge geodetic basis of G and $x \notin W$.

Then there are two vertices u_1, v_1 in W such that the shortest path between them must contain x, say P: $u_1 \ldots uxv \ldots v_1$.

Since x is an extreme vertex, u and v are adjacent. Moreover, the other path Q: $u_1 \ldots uv \ldots v_1$ exists between u_1 and v_1 having length, one less than the path P, and will not contain the edges ux and xv.

It contradicts the facts that P is the shortest path and W is an edge geodetic basis of G.

Hence $x \in W$.

1.6 Corollary. Each pendant vertex of G belongs to every vertex geodetic basis of G.

Proof. Every pendent vertex is an extreme vertex.

It completes the proof.

1.7 Theorem. For any graph, no cut vertex of G belongs to any edge geodetic basis of G.

Proof. Let x be a cut vertex of a connected graph G and $W \subseteq V$ be an edge geodetic basis of G.

Suppose $x \in W$.

Since x is a cut vertex, $G - x$ is disconnected and suppose that $G - x$ consists of k components $G_1, G_2, \ldots, G_k$ where $k \geq 2$.

Obviously W contains at least one vertex y_i (being adjacent to x) of each component G_i, $i = 1, 2, \ldots, k$.

Otherwise, W will not be the edge geodetic cover of G.

Consider the set $W \setminus \{x\}$ and any edge $u_1u_2 \in E$ where $u_1 \neq x$, $u_2 \neq x$.

Then u_1u_2 is on a shortest vw-path in G where $v, w \in W$. If $v \neq x$ and $w \neq x$, then u_1u_2 lies on a shortest vw-path where $v, w \in W \setminus \{x\}$.

Suppose $v = x$ and $w = y_i$ where $y_i \in G_i$. Then u_1u_2 is on a shortest xy_i-path, say P.

Consider the vertex y_j of G_j where $j \neq i$. Let Q be a shortest y_jx-path.

Then the union of P and Q is a shortest y_iy_j-path say R, both xu_1 and $u_1u_2$lie on R. It follows from these discussions that $W \setminus \{x\}$ is an edge geodetic cover of G.

This contradicts to the hypothesis that W is an edge geodetic basis of G.

Hence $x \notin W$.

The proofs of Theorem 1.5, Corollary 1.6 and Theorem 1.7 can also be seen in [4].

From Theorem 1.4, Theorem 1.5 and Theorem 1.7, we obtained the following theorem which states the bounds of edge geodetic numbers of a graph.

1.8 Theorem. For any graph G of order n with m extreme vertices and k cut vertices, $\max\{2, m\} \leq g_e(G) \leq n - k$.

By applying above theorems, we derive the edge geodetic numbers of some well-known graphs that are described in Theorem 1.9 to Theorem 1.13.

1.9 Theorem. The edge geodetic number $g_e(P_n)$ of any path $P_n$is 2.

Proof. Every path has two extreme vertices. Thus the number of extreme vertices of the path P_n is 2.

By Theorem 1.8, $g_e(P_n) \geq \max\ \{2, 2\}$

$= 2.$

Moreover the path P_n has $n - 2$ cut vertices.

By Theorem 1.8, $g_e(P_n) \leq n - (n - 2)$

$= 2.$

Hence, $g_e(P_n) = 2.$

1.10 Theorem. The edge geodetic number of any tree is the number of its pendant vertices.

Proof. Consider the tree T with n vertices and m pendant vertices. Every tree has at least two pendant vertices and every pendant vertex is extreme vertex. Thus $m \geq 2$.

By Theorem 1.8, $g_e(T) \geq \max\ \{2, m\}$

$= m.$

Moreover, every vertex in a tree is either pendant vertex or cut vertex.

Hence, the number of cut vertices of the tree T is $n - m$. By Theorem 1.8, $g_e(T) \leq n - (n - m)$

$= m.$

So, we get $g_e(T) = m.$

1.11 Theorem. The edge geodetic number of an even cycle is 2 and of an odd cycle is 3.

Proof. Let C be a cycle with $2n$ vertices $v_1, v_2, \ldots, v_{2n}$ in order.

Let $W = \{ v_1, v_{n+1}\}$. Then every edge of C is contained in a shortest path joining the vertices of W. Therefore W is an edge geodetic cover of C and so $g_e(C) \leq |W| = 2$. Since $g_e(C) \geq 2$, $g_e(C) = 2$.

Let C be a cycle with $2n + 1$ vertices $v_1, v_2, \ldots, v_{2n+1}$ in order. Let $W = \{ v_1, v_{n+1}, v_{n+2}\}$. Then every edge of C is contained in a shortest path joining the vertices of W. Therefore W is an edge geodetic cover of C and so $g_e(C) \leq |W| = 3$. Since $g_e(C) \geq 2$, $g_e(C) = 3$.

1.12 Theorem. For the complete graph $K_n (n \geq 2)$, $g_e(K_n) = n$.

Proof. Consider the complete graph K_n.

In any complete graph every vertex is extreme vertex and no vertex is cut vertex. By Theorem 3.11, $g_e(K_n) \geq \max \{ n, 2\}$

$$= n.$$

And $\qquad g_e(K_n) \leq n - 0$

Hence, $\qquad g_e(K_n) = n.$

1.13 Theorem. For the complete bipartite graph $G = K_{m,n}$,

$g_e(G) = 2$ if $m = n = 1$;

$g_e(G) = n$ if $m = 1, n \geq 2$;

$g_e(G) = \min \{m, n\}$ if $m, n \geq 2$.

Proof. (i) For $m = n = 1$, $K_{1,1}$ is a path P_2 and hence $g_e(K_{1,1}) = 2$.

(ii) For $m = 1$ and $n \geq 2$, consider $K_{1,n}$.

$K_{1,n}$ has n extreme vertices and only one cut vertex.

By Theorem 3.11, $\max \{n, 2\} \leq g_e(K_{1,n}) \leq (n + 1) - 1$

$$n \leq g_e(K_{1,n}) \leq n$$

$$g_e(K_{1,n}) = n.$$

(iii) Now consider $K_{m,n}$ for m, $n \geq 2$ and suppose $m \leq n$.

The $K_{m,n}$ can be decomposed as follows:

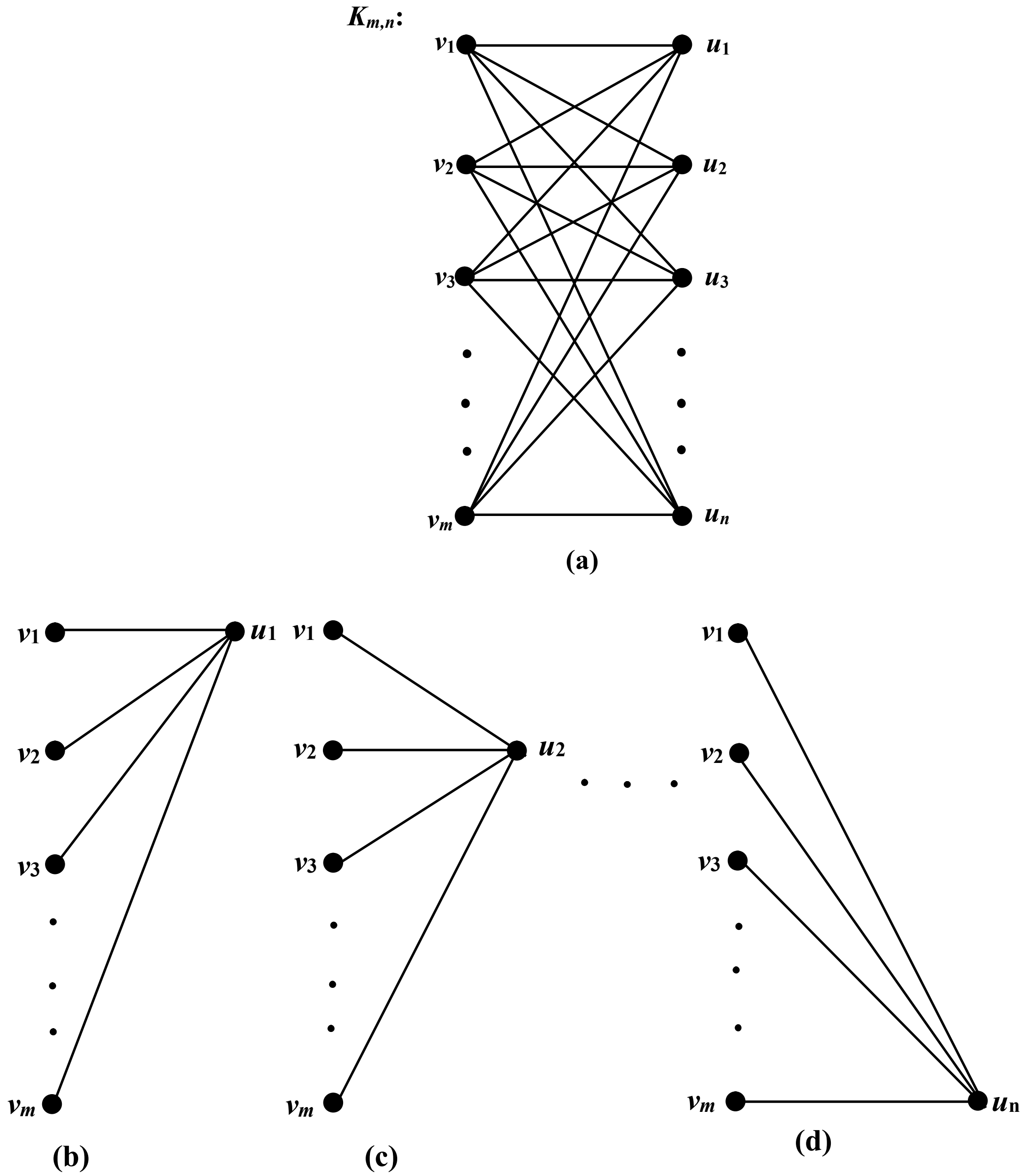


Figure. 2

Now, $g_e(K_{m,n}) = g_e(\bigcup_n K_{m,1})$

$= \text{minimum order of edge cover of } \bigcup_n K_{m,1}$

$= \text{minimum order of union of edge geodetic cover of } K_{m,1}$

$= \text{minimum order of } \bigcup_n \{ v_1, v_2, \ldots, v_m \}$

$= \text{minimum order of } \{ v_1, v_2, \ldots, v_m \}$

$= m.$

The proofs of theorems from Theorem 1.10 to Theorem 1.13 can also be seen in [4].

2. Conditions for Graphs to Have Some Specified Edge Geodetic Numbers

In this section, we study some theorems that characterize graphs for which the edge geodetic number $g_e(G)$ is 2and some necessary and sufficient conditions of a graph G with n vertices to have the edge geodetic number $g_e(G) = n - 1$ or$g_e(G) = n$. We mainly refer to [5].

2.1 Theorem. For a connected graph G, $g_e(G) = 2$ if and only if there exist peripheral vertices u and v such that every edge of G is on a diametral path joining u and v.

2.2 Theorem. If G has exactly one vertex v of degree $n - 1$, then $g_e(G)= n - 1$.

2.3 Corollary. If G has exactly one vertex v of degree $n - 1$, then G has a unique edge geodetic basis consisting of all the vertices of G other than v.

Proof. Let G be a graph with n vertices and suppose the vertex v is the only one vertex of degree $n - 1$. By Theorem 2.2,$g_e(G) = n - 1$.

Since $d(v) = n - 1$, v must be adjacent to the remaining $v_1, v_2, \ldots, v_{n-1}$ vertices of G. It means that v is on the shortest path with length 2 of any two vertices of $S = \{ v_1, v_2, \ldots, v_{n-1}\}$. In other words, all the edges of G which join v with each of $v_1, v_2, \ldots, v_{n-1}$ are on the shortest path of any two vertices of S.

On the other hand, each edge joining any two vertices from v_1, v_2, . . .,v_{n-1}does not lie on any geodesic joining two vertices of S other than themselves. Thus S is a geodetic cover and $g_e(G) \leq n - 1$. But $g_e(G) = n - 1$ and S is the geodetic basis of G. Since v is the exactly only one vertex of degree $n - 1$, S is a unique edge geodetic basis consisting of all the vertices of G other than v.

2.4 Theorem. Let G be a graph of order $n \geq 3$. If G contains a cut vertex of degree $n - 1$, then $g_e(G) = n - 1$.

Proof. Let v be a cut vertex of G of degree $n - 1$. It must be the only such vertex. For, suppose u be another cut vertex of degree $n - 1$. So, u will be adjacent with the remaining $n - 1$ vertices of G. Although we remove v from G, G will bestill connected. It contradicts that v is a cut vertex. Sov is the only vertex of degree $n - 1$ and hence by Theorem 2.3, $g_e(G) = n - 1$.

Now we discuss the edge geodetic number of a graph having more than one vertex of degree $n - 1$.

2.5 Theorem. If G has more than one vertex of degree $n - 1$, then every edge geodetic cover contains all those vertices of degree $n - 1$.

2.6 Theorem. For any graph G with at least two vertices of degree $n - 1$, $g_e(G) = n$.

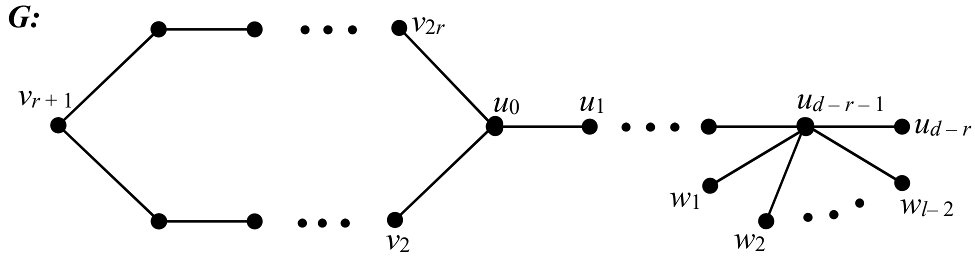
2.7 Theorem. For positive integers r, d and $l \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$, $g_e(G) = l$.

3. Constructions of Graphs with the Specified Edge Geodetic Numbers

From the previous sections, we can construct some graphs which have the specified edge geodetic numbers.

3.1 Graphs with the given the edge geodetic numbers

For any positive integer $l \geq 2$, the graphs with the $(d + r + l - 2)$ vertices have the edge geodetic number $g_e(G) = l$ where $r \leq d \leq 2r$ is as follows:



Figuer. 1

3.2 Graphs with the edge geodetic number 2

(1) For the edge geodetic number $g_e(G) = 2$, the graph is stated below.

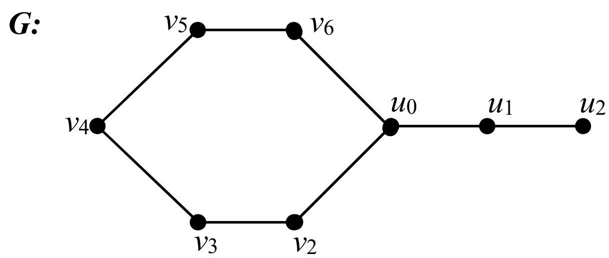


Figure. 2

(2) For the pathP_n with n vertices, the edge geodetic number is $g_e(P_n) = 2$.

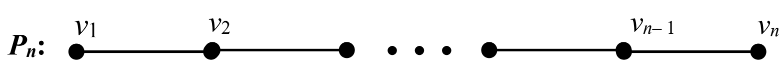


Figure. 3

(3) For the even cycle $C_{2n}$with $2n$ vertices, the edge geodetic number is $g_e(C_{2n}) = 2$.

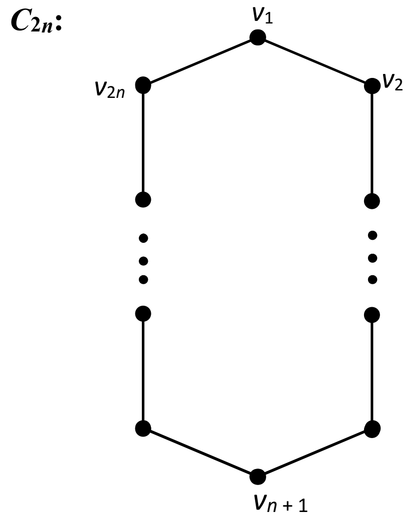
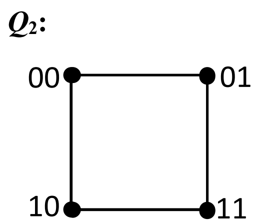


Figure. 4

(4) The cube Q_n with n vertices has the edge geodetic number is $g_e(Q_n) = 2$.



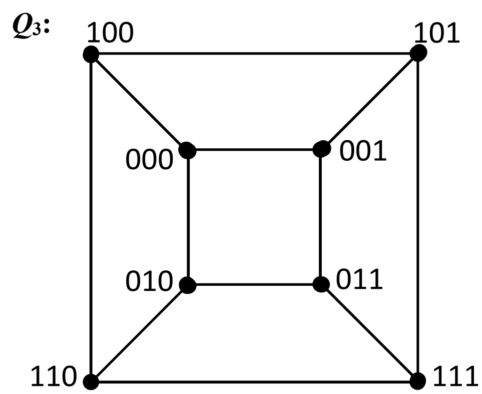


Figure. 5

3.3 Graphs with *n* vertices having the edge geodetic number*n* – 1

(1) The star with n vertices, say $K_{1,n-1}$ has the edge geodetic number $n-1$.

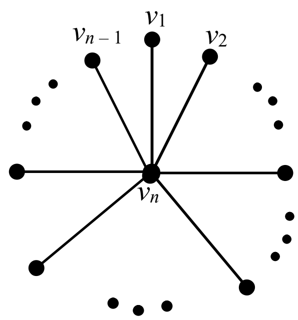


Figure. 6

(2) The wheel with n vertices, say $W_{1,n-1}$ has the edge geodetic number $n-1$.

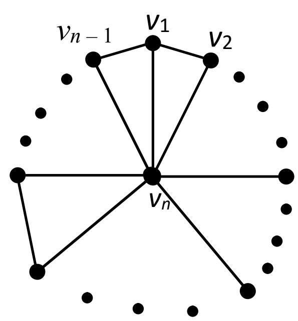


Figure. 7

3.4 Graphs with *n* vertices having the edge geodetic number *n*

(1) The complete graph, K_n has the edge geodetic number n.

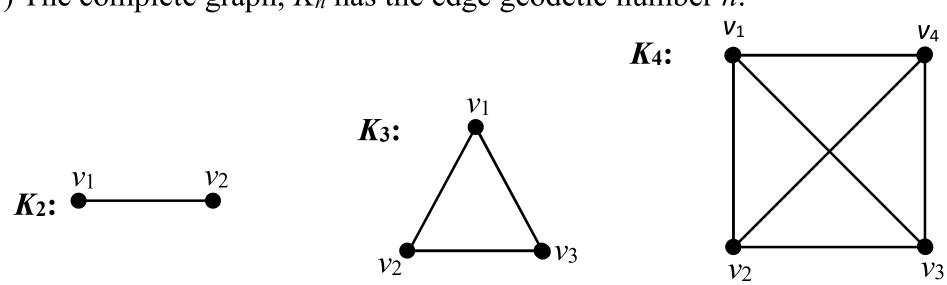


Figure. 8

References

1. M. Atici, On the edge geodetic number of a graph, *Intern. J. Computer Math*., **80**, 7 (2003), 853-861.
2. J. A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Fifth Printing, American Elsevier, New York, 1982.
3. F. Buckley and F. Harary, *Distance in Graphs*, Addison Wesley, New York, 1990.
4. A. P. Santhakumaran and J. John, Edge geodetic number of a graph, *Journal of Discrete Mathematical Sciences and Cryptography*, **10**,3 (2007), 415-432.